Target one: To estimate the tail index of return distributions (alpha).

Given a n-sample return rates,

$$X_1, X_2, \ldots, X_m, X_n \ ,$$

Taking its descending order, that is

$$X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(m)} \geq \ldots \geq X_{(n)}$$

Based on the Extreme value theory, if it is fat-tailed, we have

$$G(x) = \begin{cases} 0, x \leq 0 \\ e^{-x^{-\alpha}}, x > 0, \alpha > 0 \end{cases}$$

Here $G(x)$ is the probability that the observation $X_{(1)}$ exceeds x, in other words,

$$P(X > X_{(1)}) = 1 - G(X_{(1)}) = 1 - e^{-X_{(1)}{}^{-\hat{\alpha}}}$$

With Taylor expansion, we have

$$P(X > X_{(1)}) = 1 - e^{-X_{(1)}{}^{-\hat{\alpha}}} \approx 1 - (1 - X_{(1)}{}^{-\hat{\alpha}} + o(X_{(1)}{}^{-\hat{\alpha}})) = X_{(1)}{}^{-\hat{\alpha}}$$

The same

$$P(X > X_{(m)}) \approx X_{(m)}{}^{-\hat{\alpha}}$$

Based on the observed sequences, its empirical densities can be given by

$$P(X > X_{(m)}) \approx X_{(m)}{}^{-\hat{\alpha}} \rightarrow \frac{m}{n}$$

So choosing different orders for m, we will have different point to estimate the value α.

Beside, take the log-difference of $X_{(m)}$ and $X_{(1)}$ to eliminate n,

$$\left[\frac{X_{(1)}}{X_{(m)}}\right]^{-\hat{\alpha}} = \frac{1}{m} \Rightarrow \left[\frac{X_{(1)}}{X_{(m)}}\right]^{\hat{\alpha}} = m$$

That is

$$\log m = \hat{\alpha}(\log X_{(1)} - \log X_{(m)}),$$

(1) Here $\hat{\alpha}$ is the slope estimator.

(2)Hill (1975) the maximum likelihood estimator, following

$$\hat{\gamma}_{n,m} = \frac{1}{m-1}\sum_{i=1}^{m-1}\log X_{(i)} - \log X_{(m)}, m > 1$$

This follows that $(\hat{\gamma}_{n,m} - \gamma)\sqrt{m} \to N(0,\gamma^2)$, so the estimator $\hat{\gamma}_{n,m}$ has the standard errors $\gamma / \sqrt{m}$.

(3)Besides, we can use professor Rachev's estimator (alpha stable), the same

$$P(X > X_{(m)}) \approx C_\alpha X_{(m)}^{-\hat{\alpha}} \to \frac{m}{n} = t$$

We can regress this linear function, we call it stable estimator.

$$\log t = \log C - \alpha \log X_{(m)} + \varepsilon$$

Target two: to compute its extreme risk in the market (using estimated alpha).

Rachev showed last week

$$\hat{x}_p = X_m \left(\frac{m}{np}\right)^{\hat{\gamma}} = X_m \left(\frac{t}{p}\right)^{\hat{\gamma}}$$

Here we use in-sample X_m and its quantiles to compute the extreme risk in the market.

It follows that

$$\frac{\sqrt{m}}{\log(t/p)}\left(\frac{\hat{x}_p}{x_p} - 1\right) \to N(0,\hat{\gamma}^2)$$

So the std errors for estimated extreme risk is $\gamma \log\left(\frac{t}{p}\right) / \sqrt{m}$

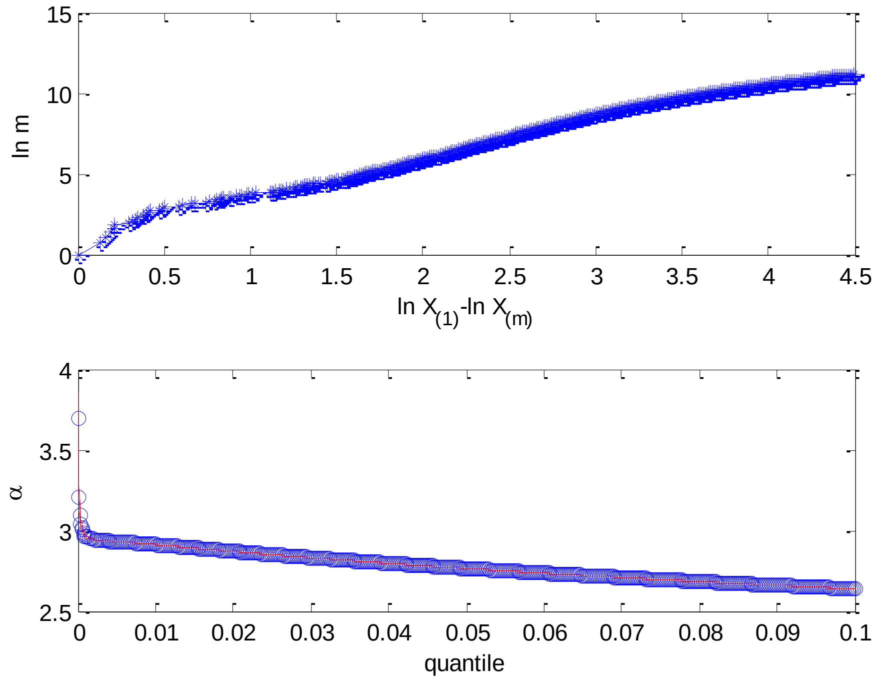
Target three: to show the evidence of scaling laws and estimate the drift exponents for the market.

Scaling laws gives a direct relation between time intervals Δt and the average volatility measured as a certain power p of the absolute returns observed over these intervals. So just regress the function:

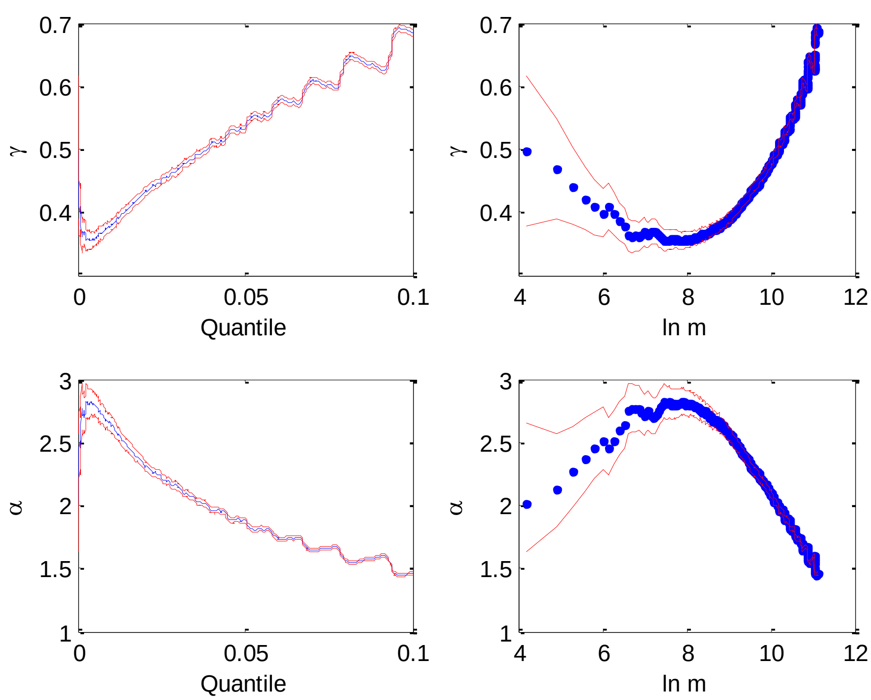
$$\{\mathrm{E}[|r|^p]\}^{1/p} = c(p)\Delta t^{D(p)}$$

$$\log\{\mathrm{E}[|r|^p]\}^{1/p} = \log c(p) + D(p)\log \Delta t$$

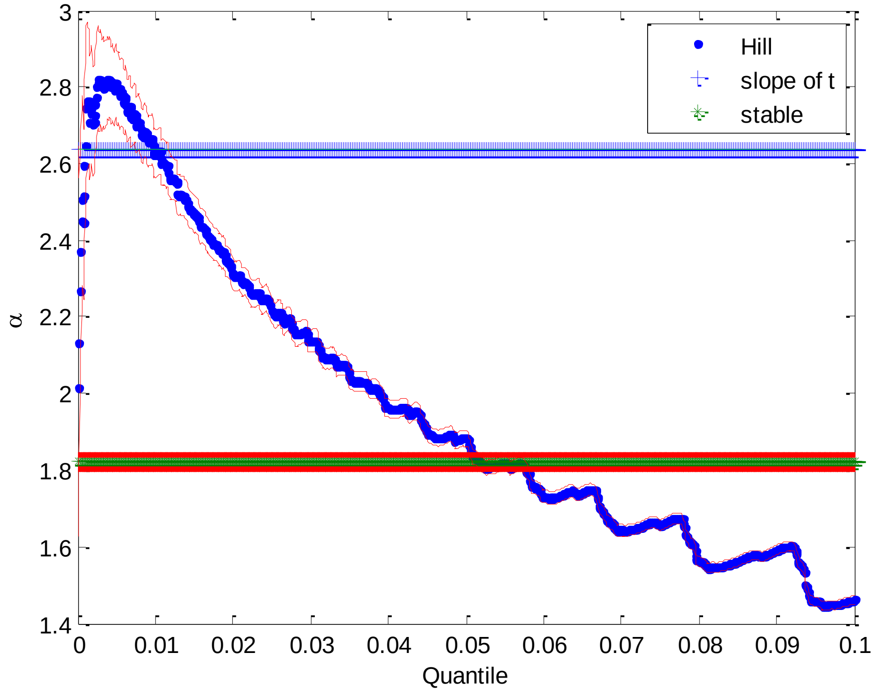
(1)Tails' slope estimator for tick-by-tick



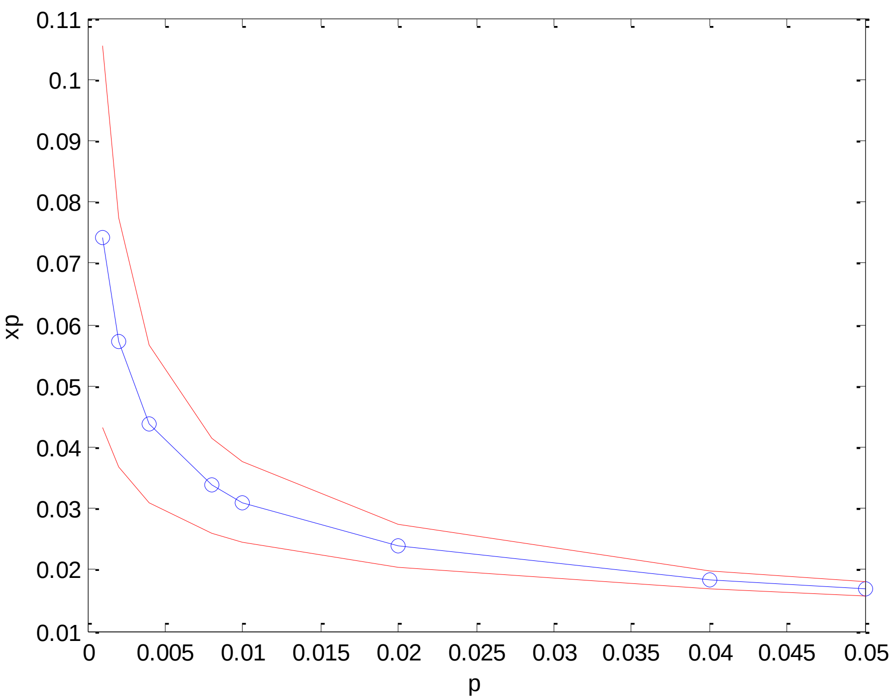
(2)Hill's estimator for tick-by-tick



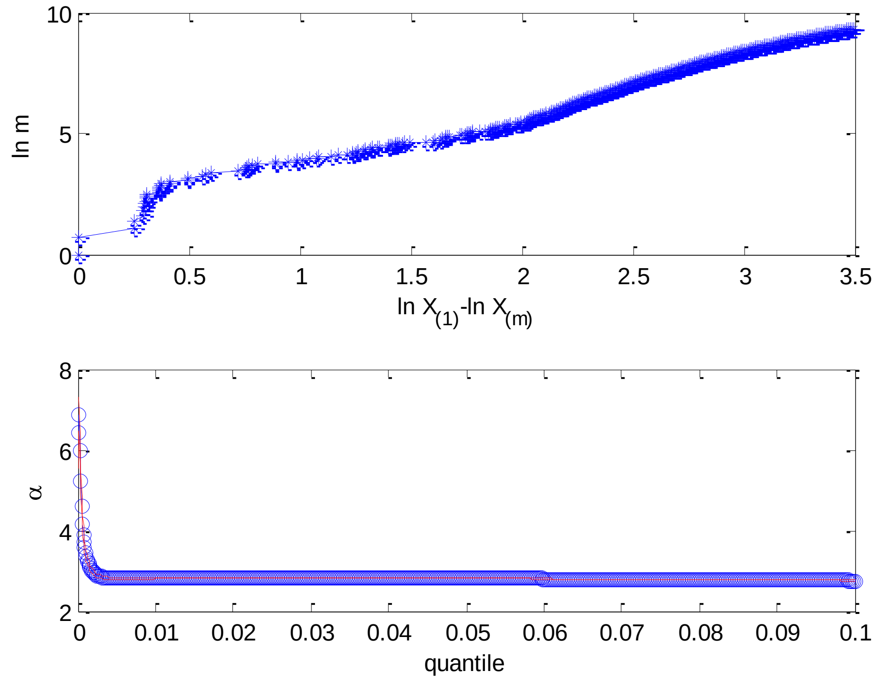
(3)Hill's, slope's and stable's estimators for tick-by-tick at level of 10% for positive tail



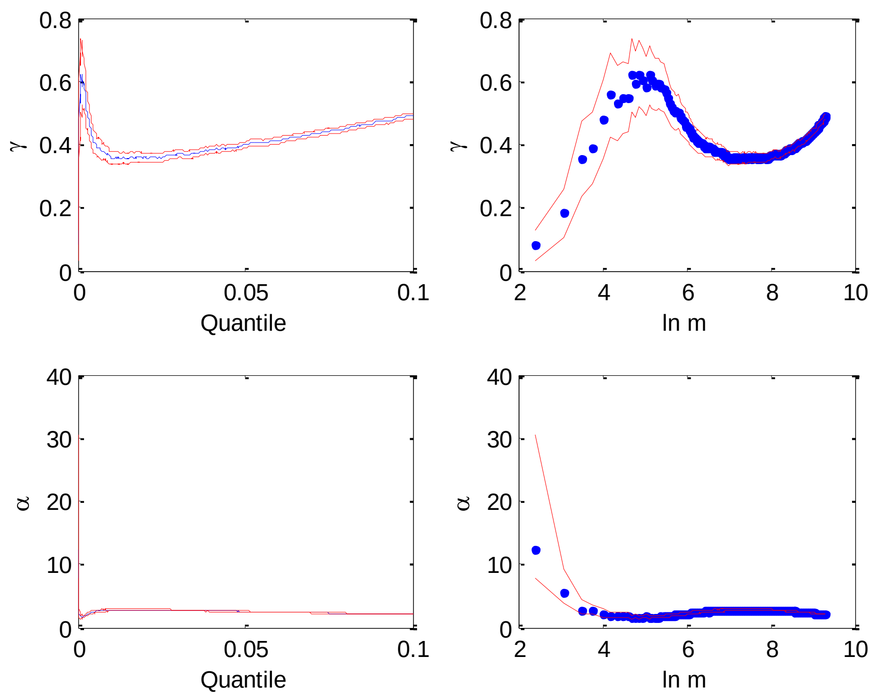
(4)Tail value for daily return rates when given the tail index from tick by tick.



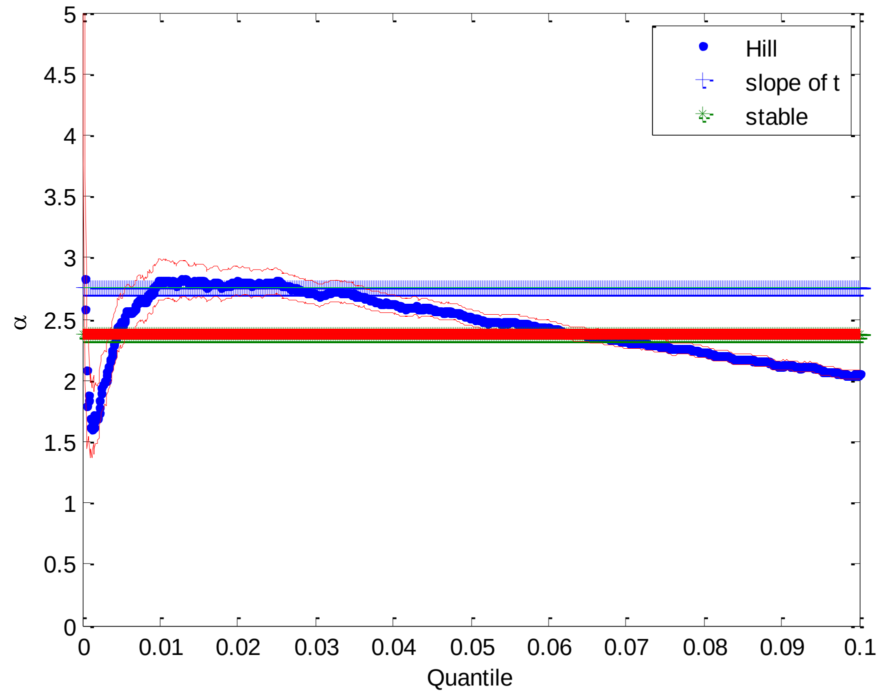
(1)Tails' slope estimator for one minute interval



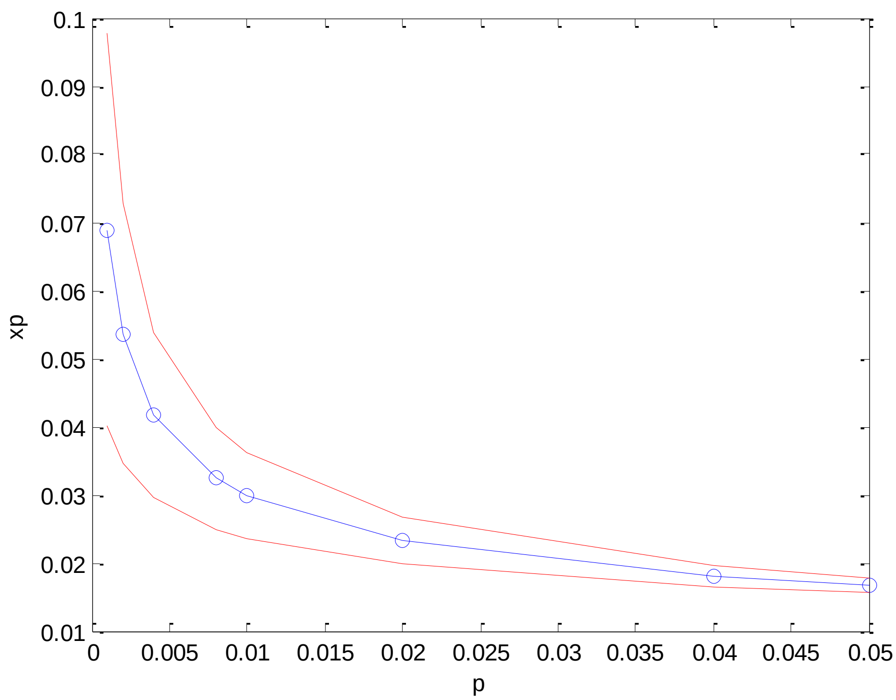
(2) Hill's estimator for one minute interval



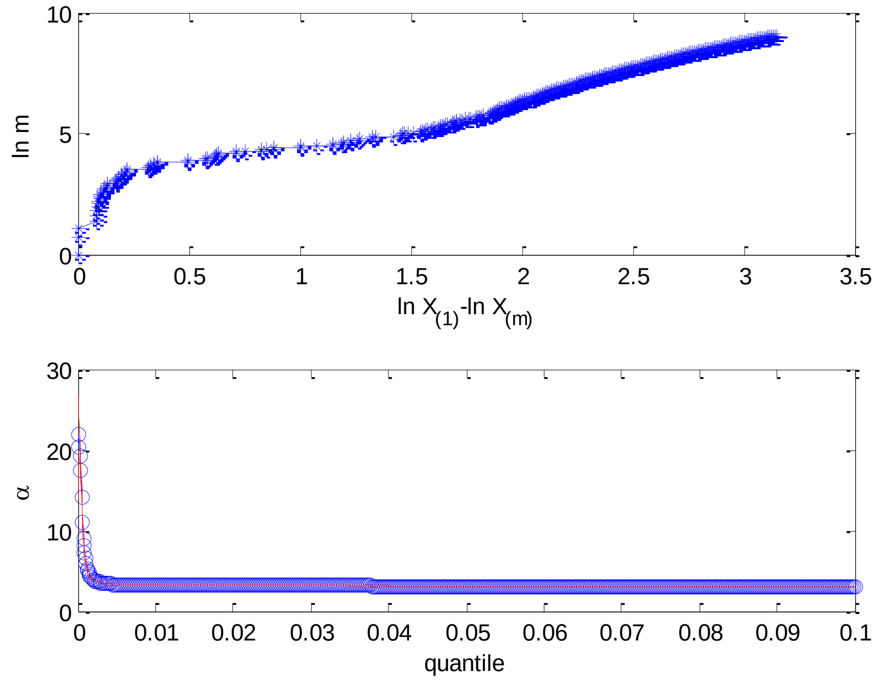
(3)Hill's, slope's and stable's estimators for one minute interval at level of 10% for positive tail



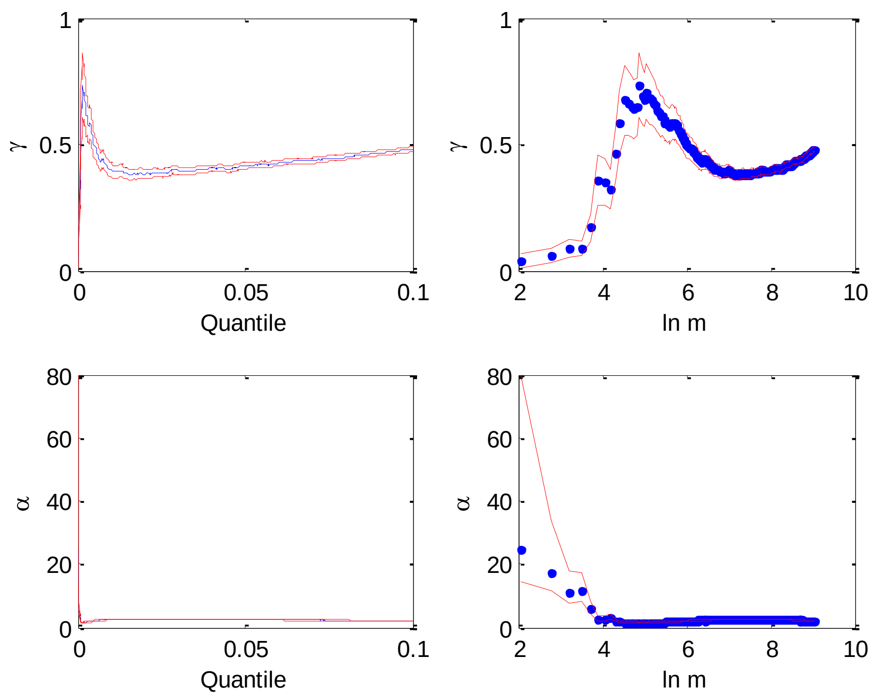
(4)Tail values for daily return rates when given the tail index from one minute interval.



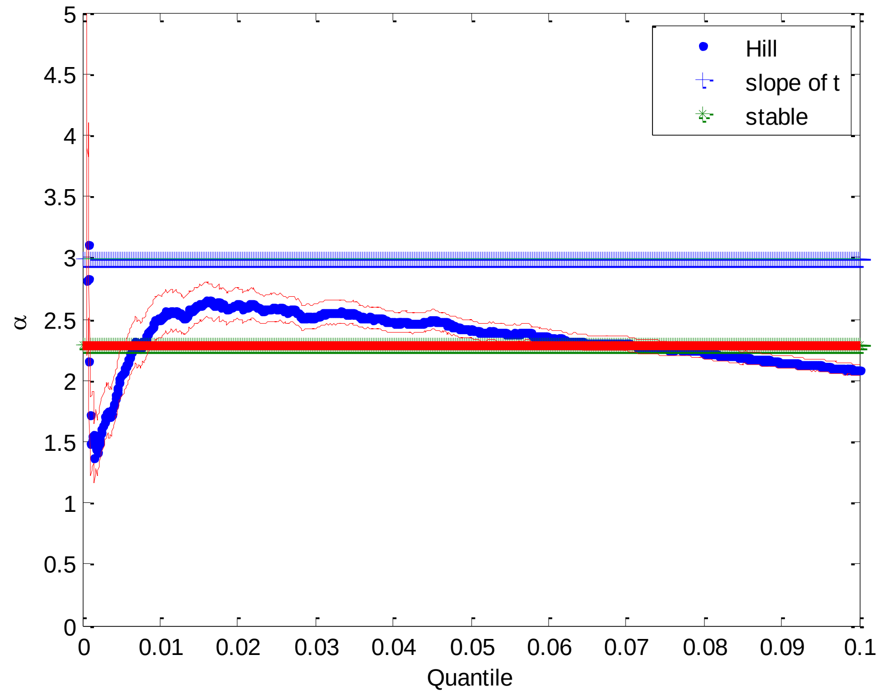
(1)Tails' slope estimator for 2 minute interval



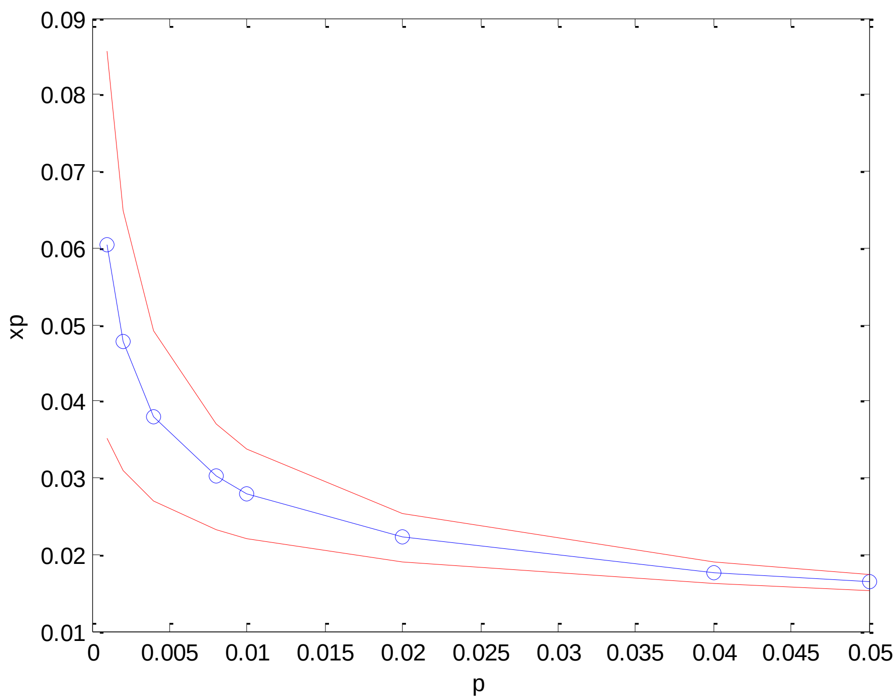
(2) Hill's estimator for 2 minute interval



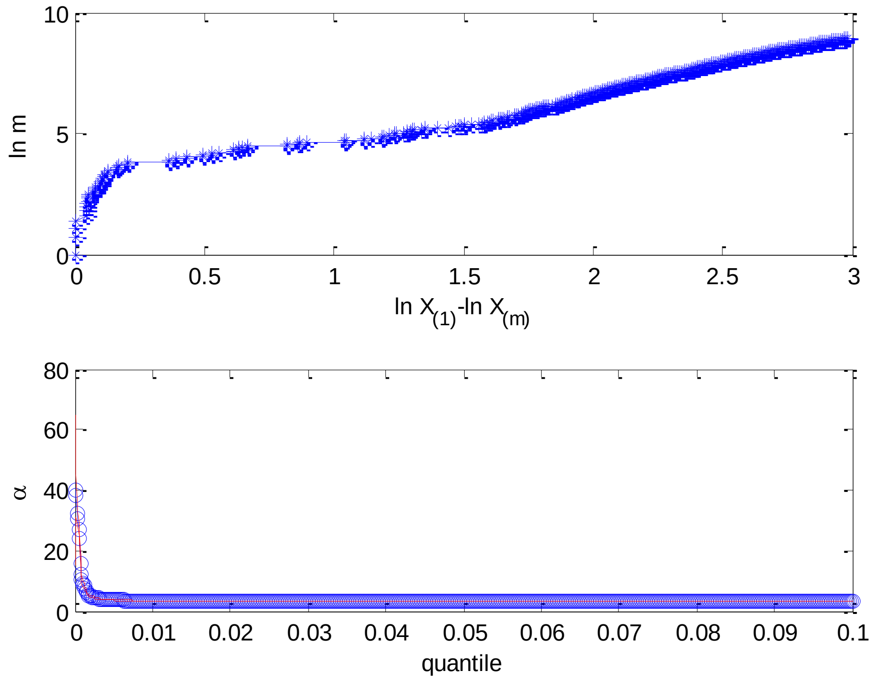
(3)Hill's, slope's and stable's estimators for 2 minute interval at level of 10% for positive tail



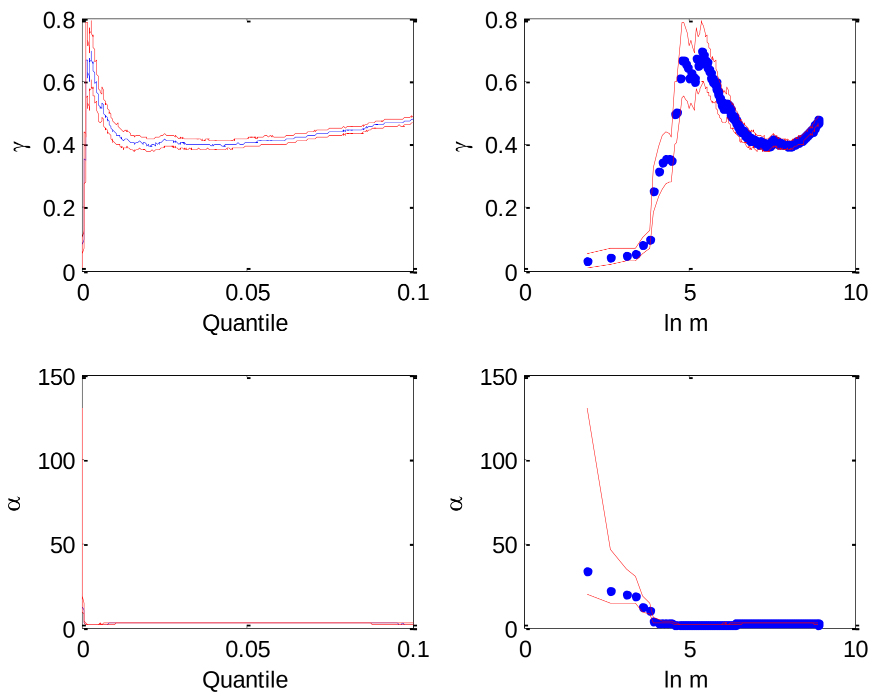
(4)Tail values for daily return rates when given the tail index from 2 minute interval.



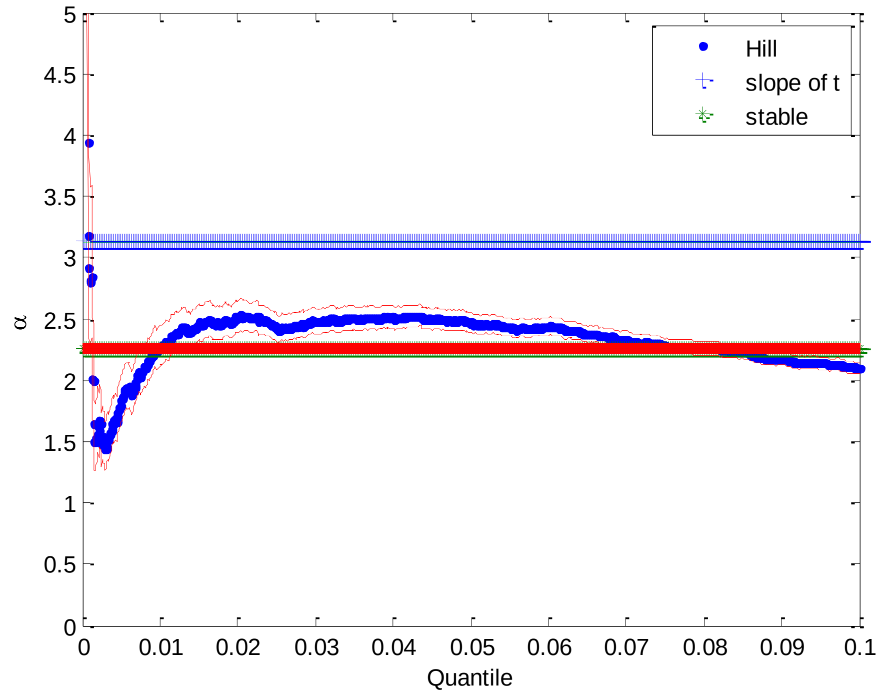
(1)Tails' slope estimator for 3 minute interval



(2) Hill's estimator for 3 minute interval



(3)Hill's, slope's and stable's estimators for 3 minute interval at level of 10% for positive tail



(4)Tail values for daily return rates when given the tail index from 3 minute interval.

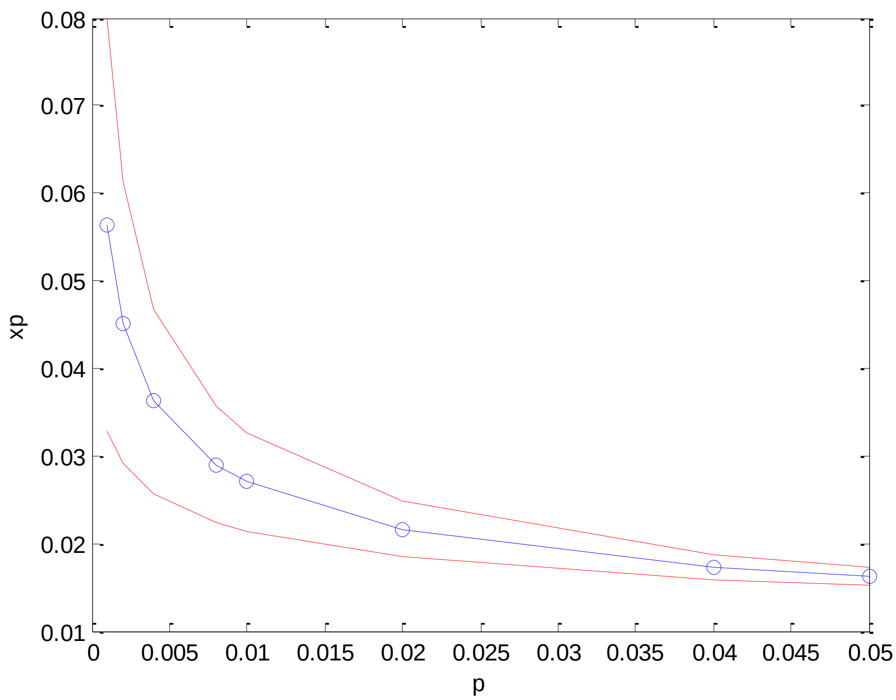


Table 1

Estimated tail index for SPX high frequency data (tail ended at 10%)

	Hill's	(Hill's std)	Tail's slope	(slope's std)	tail's alpha	(alpha' std)
ticks	1.460834	0.005623	2.6368	0.000471	1.822826	0.010398
1 min	2.047291	0.019429	2.754462	0.000606	2.381554	0.009069
2 min	2.085342	0.022826	2.991766	0.001327	2.28575	0.008637
3 min	2.093991	0.024342	3.131126	0.001812	2.264227	0.008914
4 min	2.058086	0.024619	3.208021	0.002305	2.191764	0.008256
5 min	2.075693	0.025436	3.328675	0.002922	2.141969	0.006216
6 min	1.987833	0.024614	3.4624	0.003533	2.114678	0.0059
7 min	1.970333	0.024712	3.588454	0.004016	2.127362	0.006616
8 min	1.953272	0.02463	3.631168	0.004199	2.129787	0.006352
9 min	2.141267	0.027266	3.738208	0.004534	2.160596	0.005908
10 min	2.079268	0.027743	3.76141	0.005163	2.117839	0.006821

Figure: Extreme risks in SPX markets

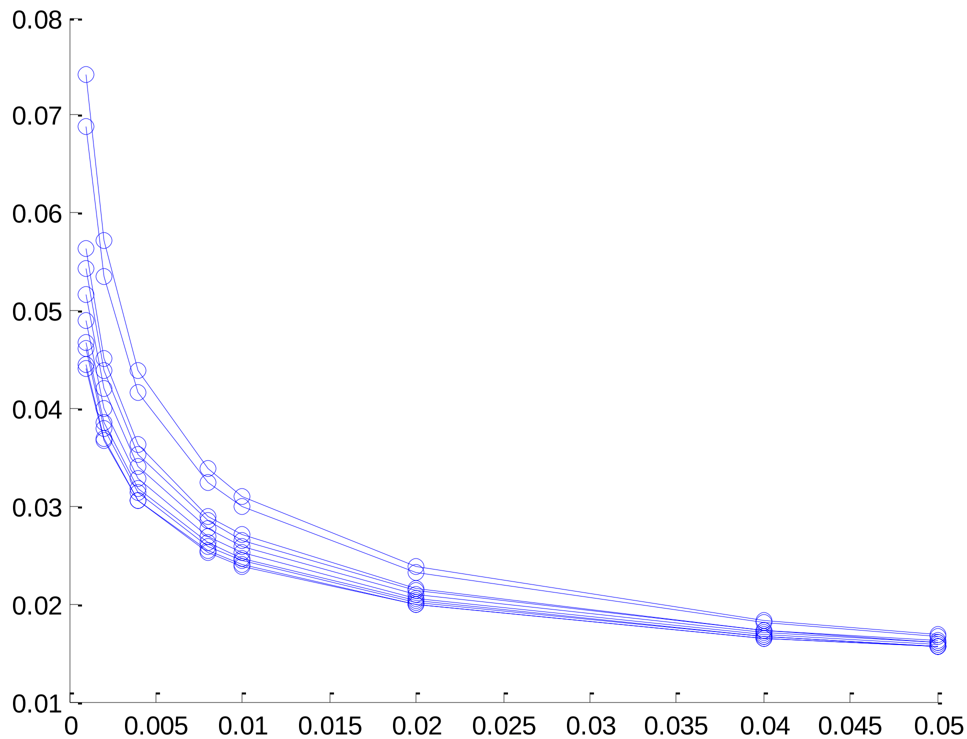


Table 2

Extreme risks in the SPX market (the event happening once in 1 week, 1 month, half year, 1 year, 2 years and 4 years), separately estimated from different time intervals from tick-by-tick to 10 minutes interval with corresponding standard errors.

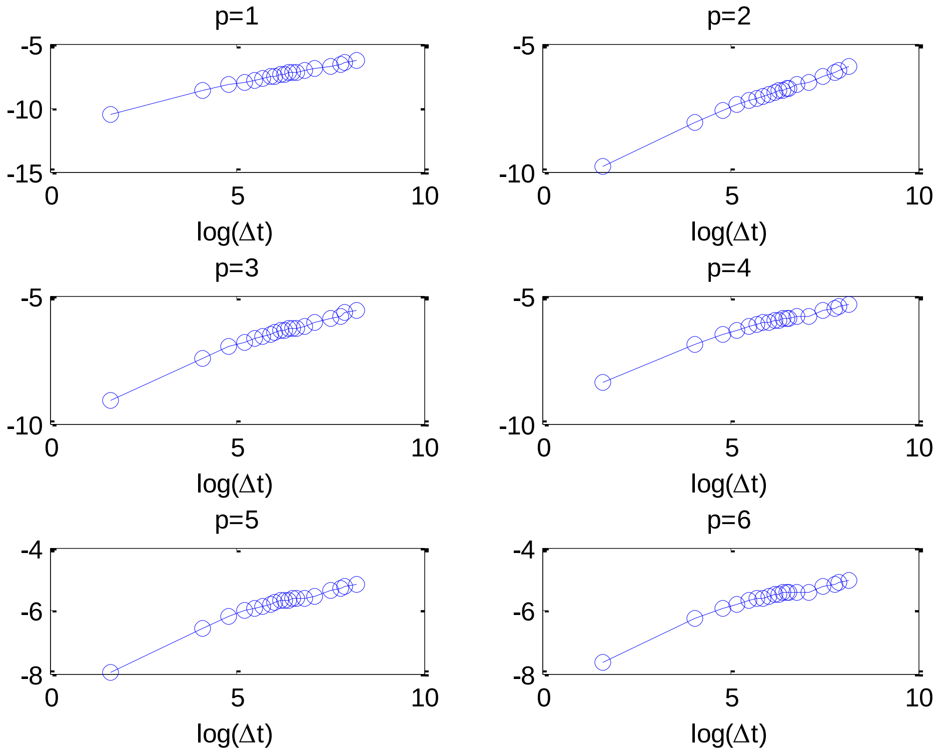
P(once) in	4 years	2 years	1 year	6 month	1 month	1 week
ticks	0.074258	0.057092	0.043895	0.033748	0.017157	0.009956
Std(ticks)	0.015859	0.010357	0.006552	0.003953	0.00059	0.00032
1 min	0.06892	0.053586	0.041665	0.032395	0.016952	0.010068
Std(1 min)	0.014718	0.009721	0.006219	0.003794	0.000583	0.000324
2 min	0.06036	0.047877	0.037976	0.030123	0.016594	0.010271
Std(2 min)	0.01289	0.008686	0.005669	0.003528	0.000571	0.00033
3 min	0.056363	0.045171	0.0362	0.029012	0.016412	0.010378
Std(3 min)	0.012037	0.008195	0.005404	0.003398	0.000565	0.000334
4 min	0.054411	0.043838	0.035319	0.028456	0.016319	0.010433
Std(4 min)	0.01162	0.007953	0.005272	0.003333	0.000561	0.000335
5 min	0.051652	0.041942	0.034058	0.027656	0.016183	0.010515
Std(5 min)	0.011031	0.007609	0.005084	0.003239	0.000557	0.000338
6 min	0.048965	0.040081	0.032809	0.026857	0.016044	0.0106
Std(6 min)	0.010457	0.007271	0.004898	0.003146	0.000552	0.000341
7 min	0.04673	0.038521	0.031755	0.026177	0.015924	0.010675
Std(7 min)	0.00998	0.006988	0.00474	0.003066	0.000548	0.000343
8 min	0.046029	0.038031	0.031422	0.025962	0.015885	0.010699
Std(8 min)	0.00983	0.006899	0.00469	0.003041	0.000547	0.000344
9 min	0.044388	0.036875	0.030634	0.02545	0.015792	0.010758
Std(9 min)	0.009479	0.00669	0.004573	0.002981	0.000543	0.000346
10 min	0.044052	0.036638	0.030472	0.025344	0.015773	0.01077
Std(10min)	0.009408	0.006647	0.004549	0.002968	0.000543	0.000346

Table 3

Drift exponents for SPX return rates. P is corresponding power; the sampling period extends half years in 2013, and includes 19 different time intervals.

	P=1	P=2	P=3	P=4	P=5	P=6
Ln(C)	-11.2654	-10.4881	-9.54692	-8.74569	-8.20048	-7.82665
Std of ln(C)	0.103512	0.098282	0.131533	0.147969	0.152025	0.151672
D	**0.613637**	**0.572097**	**0.500255**	**0.433604**	**0.389784**	**0.361166**
Std of D	0.016478	0.015645	0.020938	0.023555	0.024201	0.024144
R squares	0.989532	0.989147	0.974942	0.958501	0.946472	0.938467
F value	1606.943	1549.345	661.4149	392.6471	300.5897	259.2755
P value	2.83E-18	3.84E-18	4.75E-15	3.48E-13	3.05E-12	1E-11
var of errors	0.009771	0.008808	0.015777	0.019966	0.021076	0.020978

Figure: regression for different powers



Fumin Zhu 11/07/2013, at Stony Brook